TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 5 2016

Section 1 Calculator Free

Integration: Partial Fractions, Area, Volume, Numerical
Differentiation: Implicit, Parametric, Logarithmic

STUDENT'S NAME: SOLUTIONS

DATE: Friday 29th July **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (7 marks)

Recall: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

Evaluate the following:

(a) $\int_1^3 \frac{2e^x}{e^x - 1}dx$ [3]

$= 2\int_1^3 \frac{e^x}{e^x-1}$ ✓

$= 2\left[\ln|e^x-1|\right]_1^3$ ✓

$= 2\left(\ln(e^3-1) - \ln(e-1)\right)$

$= 2\ln\left(\frac{e^3-1}{e-1}\right)$ ✓

$= \underline{\underline{2\ln(e^2+e+1)}}$

F.Y.I.
$a^3 \pm b^3$
$= (a\pm b)(a^2 \mp ab + b^2)$

(b) $\int_0^{\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)}dx$ [4]

$= -\int_0^{\frac{\pi}{3}} \frac{-\sin x}{\cos x}dx$ ✓

$= -\left[\ln|\cos x|\right]_0^{\frac{\pi}{3}}$ ✓

$= -\left(\ln(\cos\frac{\pi}{3}) - \ln(\cos 0)\right)$

$= -\left(\ln\frac{1}{2} - \ln 1\right)$

$= -\ln\frac{1}{2} + \ln 1$ ✓ OR $\ln\left(\frac{1}{2}\right)^{-1} + 0 = \underline{\underline{\ln 2}}$

$= \ln\left(\frac{1}{\frac{1}{2}}\right)$

$= \underline{\underline{\ln 2}}$ ✓

2. (7 marks)

Determine $\int \frac{x-4}{x^2-5x+6}\,dx$

Consider: $$\frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$ ✓

$$\Rightarrow \quad x-4 = A(x-2) + B(x-3)$$

$$\Rightarrow \quad x-4 = Ax-2A + Bx-3B$$

$$\Rightarrow \quad x-4 = (A+B)x - (2A+3B)$$

$\Rightarrow \quad A+B=1$ and $2A+3B=4$

$\Rightarrow A = 1-B$ $\quad \Rightarrow 2(1-B)+3B=4$

$\Rightarrow 2-2B+3B=4$

$\therefore \underline{\underline{A=-1}}$ ✓ $\quad \therefore \underline{\underline{B=2}}$ ✓

$$\int \frac{x-4}{x^2-5x+6}\,dx = \int \frac{-1}{x-3}\,dx + \int \frac{2}{x-2}\,dx$$ ✓

$$= -\int \frac{1}{x-3}\,dx + 2\int \frac{1}{x-2}\,dx$$ ✓

$$= -\ln|x-3| + 2\ln|x-2| + C$$ ✓

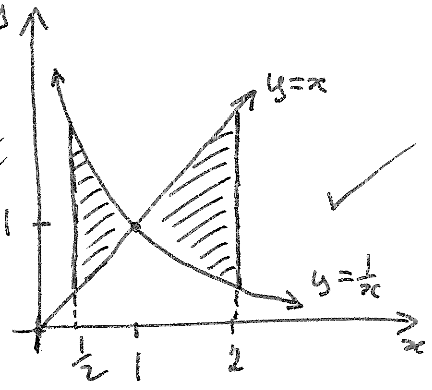
$$= \ln(x-2)^2 - \ln|x-3| + C$$

$$= \ln\frac{(x-2)^2}{|x-3|} + C$$ ✓

3. (6 marks)

Calculate the area trapped between the curves: $y = x$, $y = \dfrac{1}{x}$ and the lines $x = \dfrac{1}{2}$ and $x = 2$.

Trapped area



$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{x} - x\right) dx + \int_{1}^{2} \left(x - \frac{1}{x}\right) dx \quad \checkmark\checkmark$$

$$= \left[\ln x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} + \left[\frac{x^2}{2} - \ln x\right]_{1}^{2} \quad \checkmark$$

$$= \ln 1 - \frac{1}{2} - \left(\ln\frac{1}{2} - \frac{1}{8}\right) + 2 - \ln 2 - \left(\frac{1}{2} - \ln 1\right) \quad \checkmark$$

$$= 0 - \frac{1}{2} + \cancel{\ln 2} + \frac{1}{8} + 2 - \cancel{\ln 2} - \frac{1}{2} + 0$$

$$= 1 + \frac{1}{8}$$

$$= \underline{\underline{\frac{9}{8} \text{ units}^2}} \quad \checkmark$$

4. (7 marks)

Given the function $y = x\sin x$, differentiate by:

(a) Using the *Product Rule* [2]

$$\frac{dy}{dx} = 1.\sin x + x\cos x$$

$$= \sin x + x\cos x$$

(b) First taking the *Natural Logarithm* of both sides [5]

$$y = x\sin x$$

$$\Rightarrow \ln y = \ln(x\sin x)$$

$$\Rightarrow \ln y = \ln x + \ln \sin x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin x}.\cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + y\frac{\cos x}{\sin x}$$

$$= \frac{x\sin x}{x} + x\sin x.\frac{\cos x}{\sin x}$$

$$= \sin x + x\cos x$$ as above in part (a)

End of Questions

TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 5   2016

Section 2   Calculator Assumed

Integration: Partial Fractions, Area, Volume, Numerical
Differentiation: Implicit, Parametric, Logarithmic

STUDENT'S NAME: SOLUTIONS

DATE: Friday 29th July   **TIME:** 25 minutes   **MARKS**: 28

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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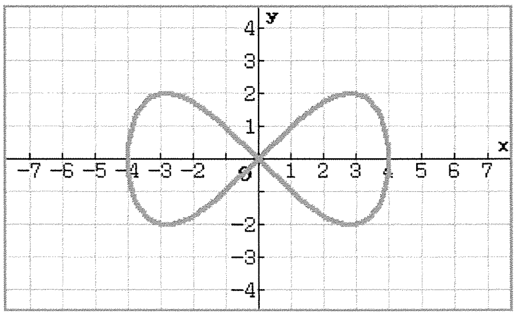
5. (8 marks)



The diagram on the right shows the curve defined parametrically as:

$$x = 4\sin(t),\ y = 2\sin(2t), \text{ for } 0 \le t \le 2\pi$$

Determine:

(a) an expression for $\frac{dy}{dx}$ in terms of t. [3]

$\frac{dx}{dt} = 4\cos t$ ✓

$\frac{dy}{dt} = 4\cos 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ ✓

$= \frac{4\cos 2t}{4\cos t}$

$= \frac{\cos 2t}{\cos t}$ ✓

(b) the coordinates and the gradient at the point when $t = \frac{\pi}{6}$. [2]

$x = 4\sin\frac{\pi}{6}$, $y = 2\sin\frac{\pi}{3}$

$= 2$, $= \sqrt{3}$

$\therefore (2, \sqrt{3})$ ✓

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}}$

$= \frac{1}{\sqrt{3}}$ ✓

$= \frac{\sqrt{3}}{3}$

(c) the exact values of t for which $\frac{dy}{dx} = 0$. [3]

$\Rightarrow \cos 2t = 0$ ✓

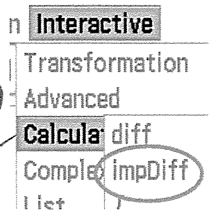
$\Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ ✓, $0 \le 2t \le 4\pi$

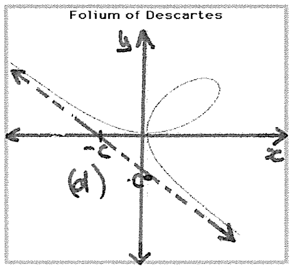
$\therefore t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ ✓, $0 \le t \le 2\pi$ as given above.

6. (11 marks)

The curve $x^3 + y^3 - 9xy = 0$, known as a *folium*, dates back to Descartes in the 1630s.

(a) Use the *implicit differentiation* utility, **impDiff**, on ClassPad to determine $\frac{dy}{dx}$.





$$\frac{dy}{dx} = \text{impDiff}\left(x^3+y^3-9xy=0\right)$$
$$= \frac{-\left(x^2-3y\right)}{y^2-3x}$$

[2]

(b) Replicate your result in part (a) by showing all the steps of implicit differentiation. [3]

$$x^3 + y^3 - 9xy = 0$$
$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - \left(9y + 9x\frac{dy}{dx}\right) = 0$$
$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{-\left(3x^2-9y\right)}{3y^2-9x}$$
$$= \frac{-\left(x^2-3y\right)}{y^2-3x}$$ as above!

(c) Determine the equation of the tangent to the curve at the point (2, 4). [2]

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{-(4-12)}{16-6} = \frac{8}{10} = \underline{\underline{\frac{4}{5}}}$$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 4 = \frac{4}{5}(x-2)$$
$$\Rightarrow \underline{\underline{y = \frac{4}{5}x + \frac{12}{5}}}$$

(d) Describe the behaviour of the curve by considering $\frac{dy}{dx}$ as x and y tend to $\pm\infty$. [2]

$\frac{dy}{dx}$ tends to $\underline{\underline{-1}}$ (as $x + y \to \pm\infty$)

Assuming symmetry about $y=x$, it appears the curve is asymptotic to a line $y=-x-c$ for some $c>0$ (see diagram above).

WORTHY OF FURTHER INVESTIGATION!

7. (6 marks)

General Statement

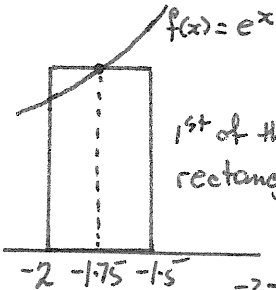
The Numerical Integration *midpoint rule* is that:

$\int_a^b f(x)\,dx \approx w\sum_{i=1}^{n} f\left(\frac{a_{i-1}+a_i}{2}\right)$, where the interval [a, b] is divided into n equal width rectangles of width w and the values $a_0, a_1, a_2, \ldots, a_n$ are the endpoints of the rectangles, so $a_0 = a$ and $a_n = b$.

Specific application

(a) Use the midpoint rule to calculate an approximation for $\int_{-2}^{2} e^x dx$ using 8 rectangles. [5]

$$\int_{-2}^{2} e^x\,dx \approx \underset{\text{width}}{\tfrac{1}{2}} \checkmark \left(e^{-1.75} + e^{-1.25} + e^{-0.75} + e^{-0.25} + e^{0.25} + e^{0.75} + e^{1.25} + e^{1.75}\right) \checkmark$$



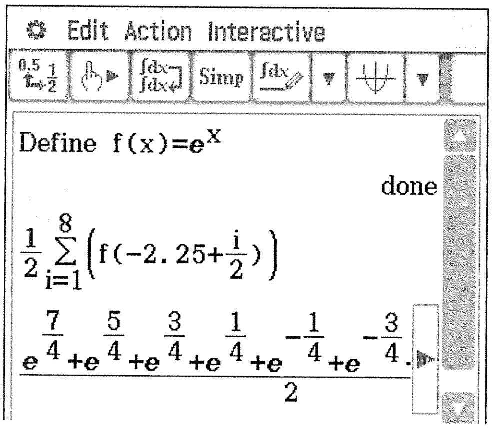
1st of the rectangles.

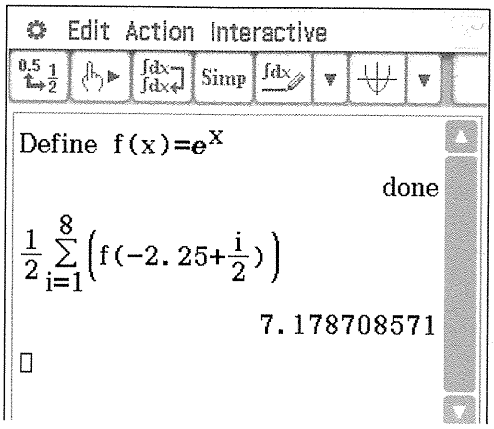
Mid point = $\frac{-2-1.5}{2}$ = -1.75

$$= \frac{1}{2}\sum_{i=1}^{8} f\left(-2.25+\frac{i}{2}\right)$$ where $f(x)$ is define as e^x.

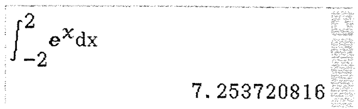
$$= 7.1787 \quad (4\,d.p)$$ ✓✓✓ (by whatever* means!)

* Are we using the technology?





(b) Compare your result to this screen capture from ClassPad.



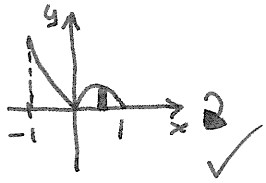
[1]

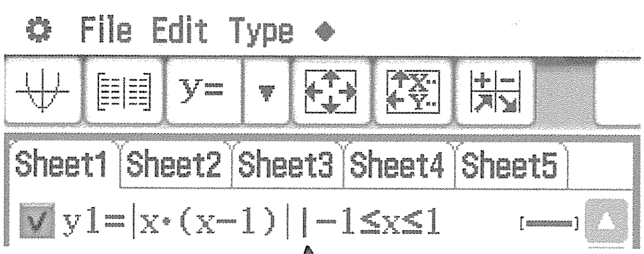
$7.1787 \approx 7.2537$ (4 d.p.)

A good comparison; would improve by increasing the number of rectangles. ✓

8. (5 marks)

Calculate the volume of solid generated when the region trapped between the curve: $y = |x(x-1)|$, the x-axis, $x = -1$ and $x = 1$ is rotated about the x-axis.





Restrict domain.

$y = |x(x-1)|$

$\Rightarrow y^2 = (x(x-1))^2$

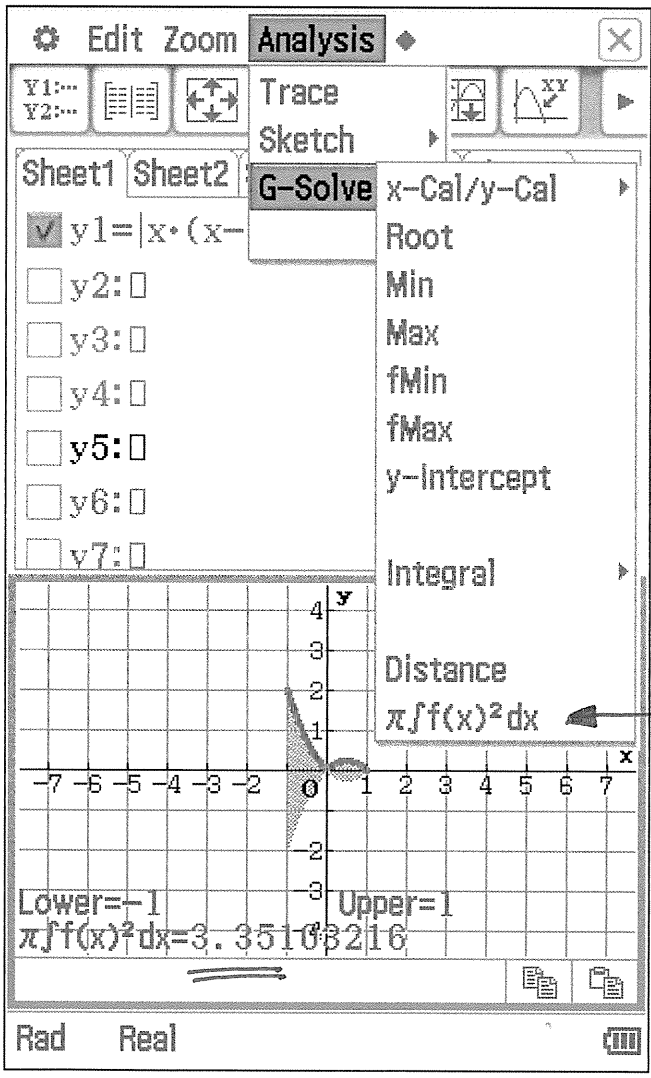
$V_x = \pi \int_{-1}^{1} (x(x-1))^2 \, dx$

By ClassPad; you could antidiff $x^4 - 2x^3 + x^2$

$= \dfrac{16\pi}{15}$

$= \underline{3{\cdot}35}$ units^3 (2d.p)

as with technology!



V_x

$\Rightarrow V_x = \underline{3{\cdot}35}$ units^3 (2d.p)

End of Questions